Linear hydromagnetic waves in an ultrarelativistic, collisionless plasma with a pressure anisotropy

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Magnetohydrodynamic (MHD) waves in a strongly magnetized collisionless plasma are considered. Consideration is carried on for the plasma with an anisotropic pressure tensor. We have derived a closed set of MHD equations describing linear hydromagnetic waves, and found the corresponding dispersion relation. We found the general solution of the dispersion equation, and considered various special cases corresponding to various standard types of MHD waves and the firehouse instability. Alternative solutions are found by means of the equations of state derived earlier for a medium with such properties and an ultrarelativistic temperature in Tsikarishvili *et al.* [Phys. Rev. A **46**, 1078 (1992)].

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I. INTRODUCTION

It is generally recognized that when a collisionless plasma is under strong enough, large-scale magnetic field its properties are essentially different along and normal to the direction of the field. In their well-known paper Chew, Goldberger and Low [1] have constructed a theory of a plasma under such conditions. They have obtained a closed set of equations for macroscopic magnetohydrodynamic (MHD) quantities describing the physical state of the medium. The model is valid for a nonrelativistic plasma when a phase velocity of disturbances $V_{\rm ph}\equiv\omega/k$ significantly exceeds the mean thermal velocity of the plasma particles [2].

In Ref. [3] we have suggested a hydrodynamical model for general-relativistic, strongly magnetized collisionless plasma with anisotropic pressure tensor. The model was the further generalization of the special-relativistic model outlined in Refs. [4, 5] which, in turn, has generalized model [1] for a nonrelativistic plasma. In Ref. [3] we have found new equations of state for the strongly magnetized collisionless plasma with *ultrarelativistic temperature* and anisotropic pressure. The qualitative difference of these equations from the corresponding ones from Chew-Goldberger-Low theory implies that various physical processes taking place in such a medium must be studied over again and the results should be principally different.

In this paper we have considered one (maybe, the most simple) example of such processes: excitation and evolution of linear MHD perturbations in a strongly magnetized collisionless plasma with anisotropic pressure. We looked at linear perturbations in the plasma with zero equilibrium flow velocity. For simplicity, we have assumed that perturbational velocity of plasma particles is nonrelativistic. However, we *do not* assume that the Alfvén speed is much smaller than the speed of light.

In the next section we have obtained the closed set of linearized equations for the perturbational quantities describing the medium under consideration. We have found an explicit dispersion equation and the general solution of this equation.

In the final section of the paper we have considered various conventional special limits of the general solution. In particular, we have found the condition for the existence of firehouse instability and dispersion relations for modified Alfvén, sound (ion-sound), and magnetosonic waves. The results are compared with the analogous ones from the nonrelativistic temperature model [1,2]. In particular, we have found that the condition for the firehouse instability remains the same, while phase velocities of the linear MHD waves drastically change. We have discussed the possible importance of these results in various astrophysical cases [magnetospheres of pulsars, jets in active galactic nuclei (AGN), stellar winds, *etc.*].

II. MAIN CONSIDERATION

Let us consider collisionless, strongly magnetized, electron-ion (or electron-positron) plasma with anisotropic pressure and infinite conductivity. The latter condition implies that

$$\vec{E} = -\vec{v}\times\vec{B} \ . \tag{2.1}$$

Note that hereafter we adopt *geometrical units* [6], in which $c = 1$. Greek indices refer to the space time and run through $0, 1, 2, 3$. The latin indices run through spatial indices $1, 2, 3$.

The plasma temperature may be arbitrarily high (even ultrarelativistic). The stress-energy tensor of the plasma is a sum of its hydrodynamical and electromagnetic parts and may be written in the following way:

$$T^{\alpha\beta} = S_1 U^\alpha U^\beta + S_2 \eta^{\alpha\beta} + S_3 h^\alpha h^\beta \ , \tag{2.2}$$

where

$$S_1 \equiv e + P_\perp + |h|^2/4\pi \ , \tag{2.3a}$$

$$S_2 \equiv P_\perp + |h|^2/8\pi \ , \tag{2.3b}$$

$$S_3 \equiv -1/4\pi + (P_\parallel - P_\perp)/|h|^2 \ . \tag{2.3c}$$

In (2.1)–(2.3) all notations are standard. In particular, e is the proper mass-energy density; $P_\parallel$ and $P_\perp$ are the plasma pressure components along and normal to the direction of the magnetic field $\vec{B}$, respectively; U^α are components of the plasma 4-velocity; $\eta^{\alpha\beta}$ is the Minkowski (flat) space-time metric tensor [6], and h^α is Lichnerowicz's 4-vector of the magnetic field [7], defined as

$$h^\alpha \equiv (\delta^\alpha_m + U^\alpha U_m) B^m/\gamma \ , \tag{2.4}$$

where $\gamma \equiv (1-v^2)^{-1/2}$, $U^t = \gamma$, and $U^i = \gamma v^i$.

In the present consideration the regular (equilibrium) velocity of the plasma is assumed to be zero, while its perturbation component is assumed to be nonrelativistic. Such assumption seems to be inevitable if we require consideration of *linear* waves. It is obvious that when perturbational components of velocity are *strongly* relativistic linear approximation ceases to be correct, since $(v/c)^2 \approx (v/c)$. That is why we will retain in our equations only those terms which are linear with respect to $\vec{v}$.

If we introduce the projection tensor in a standard way:

$$P^{\alpha\beta} \equiv \eta^{\alpha\beta} + U^\alpha U^\beta, \tag{2.5}$$

then the general equation of motion for the plasma may be written as

$$P_i{}^\beta T_\beta{}^\gamma{}_{;\gamma} = 0, \tag{2.6}$$

which further reduces (by taking into account all above mentioned assumptions) to the following equation:

$$S_1 \partial_t \vec{v} + S_3(\partial_t \vec{v}, \vec{B})\vec{B} + \mathrm{grad} S_2 + (\vec{B}, \nabla)(S_3 \vec{B}) = 0. \tag{2.7}$$

Together with the continuity equation

$$\partial_t \rho + \mathrm{div}(\rho \vec{v}) = 0 \tag{2.8}$$

and the induction equation

$$\partial_t \vec{B} = (\vec{B}\nabla)\vec{v} - (\vec{v}\nabla)\vec{B} - \vec{B}\,\mathrm{div}\vec{v}, \tag{2.9}$$

(2.7) constitutes the complete set of differential equations. Note that (2.7) reduces to the conventional MHD Euler equation for isotropic pressure ($P_\perp = P_\parallel = P$) and the the nonrelativistic temperature ($e \approx \rho$) case, as it, certainly, should.

For simplicity, in this paper we shall consider the two-dimensional problem (generalization to three dimensions seems to be straightforward). Let us suppose that a regular (unperturbed) state of the medium is characterized by the regular magnetic field $\vec{B}_0$, directed along the Y axis; and the following values of density and pressure: ρ_0, $P_{\parallel 0}$, and $P_{\perp 0}$. Under such conditions we can carry out Fourier expansion for any quantity f appearing in our equations:

$$f = f_0 + f' \exp[i(\vec{k}\cdot\vec{x} - \omega t)] \ . \tag{2.10}$$

Equations of state allow us to specify the value of proper mass-energy density e. When the plasma temperature is nonrelativistic, we can use the well-known expressions [2,1,8]

$$e = \rho + P_\perp + P_\parallel/2, \tag{2.11a}$$

$$P_\perp = C_\perp \rho |\vec{B}| \ , \tag{2.11b}$$

$$P_\parallel = C_\parallel \rho^3 |\vec{B}|^{-2}, \tag{2.11c}$$

while when the temperature is ultrarelativistic these expressions are replaced by the ones, originally obtained in [3],

$$e = 2P_\perp + P_\parallel \ , \tag{2.12a}$$

$$P_\perp = C_\perp \rho \sqrt{|\vec{B}|} \ , \tag{2.12b}$$

$$P_\parallel = C_\parallel \rho^2 |\vec{B}|^{-1}. \tag{2.12c}$$

The equations of state (2.11) and (2.12) may be represented by the following *one* pair of expressions [8]:

$$P_\perp = C_\perp \rho |\vec{B}|^\delta, \tag{2.13a}$$

$$P_\parallel = C_\parallel \rho \left(\frac{\rho}{|\vec{B}|} \right)^{2\alpha} . \tag{2.13b}$$

Evidently, (2.11) correspond to the case $\delta = \alpha = 1$, while ultrarelativistic equations of state (2.12) correspond to the case $\delta = \alpha = 1/2$. Note that in (2.11)-(2.13) $C_\parallel$ and $C_\perp$ are some constants. Note also that we have written $|\vec{B}|$ instead of $|h|$ due to the linear approximation restraints.

Let us introduce the following dimensionless perturbational functions: $d \equiv \rho'/\rho_0$, $b_x \equiv B'_x/B_0$, $b_y \equiv B_y{}'/B_0$, $S_\perp \equiv P_\perp{}'/P_{\perp 0}$, $S_\parallel \equiv P_\parallel{}'/P_{\parallel 0}$. Dimensionless pressure perturbations $S_\parallel$ and $S_\perp$ may be expressed through perturbations of density and magnetic field by help of equations of state. As a result, we get

$$S_\perp \equiv d + \delta b_y, \tag{2.14a}$$

$$S_\parallel \equiv (2\alpha + 1)d - 2\alpha b_y \ . \tag{2.14b}$$

Now we can write Eqs. (2.7)–(2.9) in dimensionless notations for perturbed quantities using Fourier expansion (2.10). The set of equations reduces to the following algebraic equations:

$$-\omega d + k_x v_x + k_y v_y = 0, \tag{2.15a}$$

$$P_{\perp 0} k_x d - S_{10}\omega v_x + L b_x = 0, \tag{2.15b}$$

$$(2\alpha+1)P_{\|0} k_y d - (e_0 + P_{\|0})\omega v_y + M b_x = 0, \tag{2.15c}$$

$$-k_y v_x - \omega b_x = 0, \tag{2.15d}$$

where $S_{10} = e_0 + P_{\perp 0} + B_0^2/4\pi$; and

$$L \equiv [(P_{\|0} - P_{\perp 0})k_y^2 - (B_0^2/4\pi)k^2 - \delta P_{\perp 0} k_x^2]/k_y, \tag{2.16}$$

$$M \equiv [(2\alpha+1)P_{\|0} - P_{\perp 0}]k_x. \tag{2.17}$$

It is obvious that the corresponding dispersion equation may be obtained from $\det(\mathbf{R}) = 0$, where $\mathbf{R}$ is the following fourth order square matrix:

$$\mathbf{R} = \begin{pmatrix} -\omega & k_x & k_y & 0 \\ P_{\perp 0} k_x & -S_{10}\omega & 0 & L \\ (2\alpha+1)P_{\|0} k_y & 0 & -(e_0 + P_{\|0})\omega & M \\ 0 & -k_y & 0 & -\omega \end{pmatrix}. \tag{2.18}$$

Introducing the useful notations $tg\theta \equiv k_x/k_y$, $k \equiv \sqrt{k_x^2 + k_y^2}$, and

$$a_1 \equiv B_0^2/4\pi + P_{\perp 0} - P_{\|0}, \tag{2.19a}$$

$$a_2 \equiv e_0 + P_{\|0}, \tag{2.19b}$$

$$a_3 \equiv (2\alpha+1)P_{\|0}\cos^2\theta, \tag{2.19c}$$

$$a_4 \equiv a_1 + (\delta P_{\perp 0} + P_{\|0})\sin^2\theta, \tag{2.19d}$$

we can write the dispersion equation in the following way:

$$A(\omega/k)^4 + C(\omega/k)^2 + D = 0\ , \tag{2.20}$$

where

$$A \equiv a_2(a_1 + a_2)\ , \tag{2.21a}$$

$$C \equiv -a_3(a_1 + a_2) - a_2 a_4\ , \tag{2.21b}$$

$$D \equiv a_3 a_4 - {P_{\perp 0}}^2 \sin^2\theta\cos^2\theta. \tag{2.21c}$$

The general solution of the dispersion relation may be written explicitly as

$$\omega^2 = \frac{k^2}{2a_2(a_1+a_2)}\left\{a_3(a_1+a_2) + a_2 a_4 \pm \sqrt{[a_3(a_1+a_2) - a_2 a_4]^2 + 4a_2(a_1+a_2){P_{\perp 0}}^2\sin^2\theta\cos^2\theta}\right\}. \tag{2.22}$$

Let us note that an angle θ, introduced above, is the angle between the direction of the wave vector $\vec{k}$ and the vector of regular magnetic field $\vec{B}_0$.

III. DISCUSSION

First of all, let us note that, as it is clearly seen from (2.22), unstable modes appear if the numerator of the right hand side of the equation is negative: $-C - \sqrt{C^2 - 4AD} < 0$. This condition, as it may easily be checked out, implies the following condition:

$$\sin^2\theta < \frac{(2\alpha+1)P_{\|0}[P_{\|0} - P_{\perp 0} - B_0^2/4\pi]}{(2\alpha+1)P_{\|0}(\delta P_{\perp 0} + P_{\|0}) - P_{\perp 0}^2}. \tag{3.1}$$

Note that for the medium with nonrelativistic temperature ($\alpha = \delta = 1$) (2.24) reduces to the well-known result [2]

$$\sin^2\theta < \frac{3P_{\|0}[P_{\|0} - P_{\perp 0} - B_0^2/4\pi]}{3P_{\|0}(P_{\perp 0} + P_{\|0}) - P_{\perp 0}^2}, \tag{3.2a}$$

while when the temperature is ultrarelativistic, and $\alpha = \delta = 1/2$, it leads to the new condition

$$\sin^2\theta < \frac{2P_{\|0}[P_{\|0} - P_{\perp 0} - B_0^2/4\pi]}{P_{\|0}P_{\perp 0} + 2P_{\|0}^2 - P_{\perp 0}^2}. \tag{3.2b}$$

Turning our attention to the stable modes, let us consider, originally, waves propagating along the magnetic field $\vec{B}_0$ ($\theta = 0$). When plasma temperature is nonrelativistic, $\alpha = \delta = 1$, (2.11) holds and our general solution (2.22) leads to the following pair of expressions:

$$\omega^2 = k^2 \frac{{B_0}^2/4\pi + P_{\perp 0} - P_{\|0}}{\rho_0 + 2P_{\perp 0} + P_{\|0}/2 + {B_0}^2/4\pi}\ , \tag{3.3}$$

$$\omega^2 = k^2 \frac{3P_{\|0}}{\rho_0 + P_{\perp 0} + 3P_{\|0}/2 + {B_0}^2/4\pi}\ . \tag{3.4}$$

The first solution describes hydromagnetic waves in strongly magnetized sparse plasma. If plasma pressure is isotropic ($P_{\perp 0} = P_{\|0} = P_0$) and also $P_0/\rho_0 \ll 1$ these are plain Alfvén waves:

$$\omega_{is} = kV_A/\sqrt{1 + V_A^2}\ , \tag{3.5}$$

where $V_A \equiv B_0/\sqrt{4\pi\rho_0}$. This dispersion relation for Alfven waves differs from the conventional one by the factor $(1 + V_A^2)^{-1/2}$. The factor arises due to the inertia corresponding to the energy density of the magnetic field [9] (or, in other words, when one takes into account the displacement current when expressing Lorentz force in conventional equation of motion). Its appearance is assured by the second term on the right hand side of (2.7) (general equation of motion). It must be emphasized that we do not assume that the Alfvén speed is much smaller than the speed of light. Note that for large enough densities, when $V_A \ll 1$ (3.5) reduces to plain Alfvén waves, while when $V_A \gg 1$ they turn smoothly into obvious electromagnetic waves in vacuum.

When plasma pressure is anisotropic but the rest mass energy dominates over thermal and magnetic energies (3.3) reduces to the well-known dispersion relation for modified Alfvén waves in anisotropic plasma [2].

When $P_{\|0} > P_{\perp 0} + B_0^2/4\pi$ the perturbations described by (3.3) become unstable— increase or decrease exponentially, $d \sim \exp(\pm\omega_1 t)$ where $\omega_1 = \mathrm{Im}(\omega)$. This is a so called firehouse instability. The criterion of its existence is

$$P_{\|0} > P_{\perp 0} + \frac{B_0^2}{4\pi}. \qquad (3.6)$$

Evidently, this criterion may also be derived directly from (3.1), by taking $\theta = 0$ in it.

The second solution (3.4) describes ion-sound waves propagating along the magnetic field.

These results, as we already noted, are widely known. Now let us see how they change in the case when plasma temperature is ultrarelativistic. In this case $\alpha = \delta = 1/2$, (2.12) holds, already and for the case ($\theta = 0$) instead of (3.3) and (3.4) we get from (2.22)

$$\omega^2 = k^2 \frac{[B_0^2/4\pi + P_{\perp 0} - P_{\|0}]}{[B_0^2/4\pi + 3P_{\perp 0} + P_{\|0}]}, \qquad (3.7)$$

$$\omega^2 = k^2 \frac{P_{\|0}}{P_{\perp 0} + P_{\|0}} \ . \qquad (3.8)$$

First of all, let us check out what happens with the condition for firehouse instability (3.6). From (3.7) we see that in this case the condition remains the same. Generally speaking, this result must be evident from the very beginning, since the condition for the instability existence most generally is $D < 0$ when $\theta = 0$. From (2.22) we see that when $\theta = 0$ dependence on parameters α and δ is canceled out. Thus we can conclude that the criterion for firehouse instability (3.6) is independent of the equation of state.

Otherwise, (3.7) and (3.8) differ significantly from the corresponding expressions for the plasma with nonrelativistic temperature. It means that in ultrarelativistic, collisionless, and strongly magnetized plasma MHD waves along the magnetic field propagate with different phase velocities. In particular, ion-sound waves in this case have a genuinely simple expression [see (3.8)] for phase velocity $V_{\mathrm{ph}} \equiv \omega/k$. When the plasma pressure is isotropic the phase velocity of these waves does not depend on any parameters of plasma (such as value of the regular magnetic field B_0, or density ρ_0), is constant, and is equal to $V_{\mathrm{ph}} = \sqrt{2}/2$. For the strongly anisotropic plasma with $P_{\|0} \gg P_{\perp 0}$ [10] $V_{\mathrm{ph}} \to 1$, while in the opposite case [11] $P_{\|0} \ll P_{\perp 0}$, we have $V_{\mathrm{ph}} \to 0$.

We have also the same situation for the waves propagating in the direction normal to the direction of magnetic field ($\theta = \pi/2$). In this particular case, in nonrelativistic limit, the solution of the dispersion equation reduces to the following form:

$$\omega^2 = 2k^2 \frac{(B_0^2/8\pi + P_{\perp 0})}{\rho_0 + 2P_{\perp 0} + P_{\|0}/2 + {B_0}^2/4\pi}. \qquad (3.9)$$

The result is also very well known (for example, see Ref. [2]). These are magnetosonic waves. Evidently they will also exist in the ultrarelativistic temperature limit. But in this case using (2.12) instead of (2.11) we get a noticeably different result:

$$\omega^2 = k^2 \frac{B_0^2/4\pi + 3P_{\perp 0}/2}{B_0^2/4\pi + P_{\|0} + 3P_{\perp 0}} \ . \qquad (3.10)$$

Comparing (3.10) with (3.9) we can definitely notice that the phase velocity of magnetosonic waves in the collisionless plasma with ultrarelativistic temperature is much more than the same velocity in the nonrelativistic temperature case (as it, certainly, should be in accordance with basic physical grounds).

Thus comparing our results with the ones from standard theory of linear MHD waves in cold collisionless plasma we find out that all existing kinds of MHD waves have in the case of ultrarelativistic temperatures definitely different phase velocities, while the condition for firehouse instability remains the same. It is also proved that the latter condition is wholly independent of the concrete kind of equations of state.

The study of the hydrodynamics of collisionless plasma in strong magnetic field is important for a wide class of astrophysical objects, including relativistic winds of pulsars and accretion discs in stellar close binary systems and jets (bipolar or unipolar outflows of plasma) in active galactic nuclei and quasars. Consideration of relativistic effects in these objects often becomes necessary not due to the relativistic velocities of plasma macroscopic (regular) motion, but when the velocities of the plasma particles' microscopic motions are high enough, i.e., when the temperature of plasma becomes ultrarelativistic. Besides, in some of these astrophysical plasma flows pressure is not a scalar. For example, in pulsar winds, due to synchrotron-radiation losses, $P_{\|} \gg P_{\perp}$ [10], while in so called "cosmic pinches" [11] $P_{\|} \ll P_{\perp}$.

That is why we think that the results obtained in this paper may be applied to various kinds of astrophysical flows where the presence of a medium with the above mentioned qualities is proved in one way or another.

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